



**FURYE O‘ZGARTIRISHLARI YORDAMIDA TELEGRAF TENGLAMASI  
UCHUN BOSHLANG‘ICH-CHEGARAVIY  
MASALANI YECHISH**

**РЕШЕНИЕ НАЧАЛЬНО-КРАЕВОЙ ЗАДАЧИ ДЛЯ ТЕЛЕГРАФНОГО  
УРАВНЕНИЯ С ИСПОЛЬЗОВАНИЕМ ПРЕОБРАЗОВАНИЯ ФУРЬЕ  
SOLVING THE INITIAL-BOUNDARY VALUE PROBLEM FOR THE  
TELEGRAPH EQUATION USING THE FOURIER TRANSFORM**

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**Annotatsiya.** Ushbu maqolada birinchi chorakda telegraf tenglamasi uchun boshlang‘ich-chegaraviy masalani Furye integral almashtirishlari usuli yordamida yechish masalasi tadqiq qilingan. Furyening kosinus almashtirishidan foydalanib, masala oddiy differensial tenglamaga keltirilgan va ikki xil fizik hol (tebranuvchi va tebranishsiz so‘nish) uchun analitik yechimlar aniq formulalar shaklida olingan. Berilgan masalaning yechimi yagonaligi isbotlangan.

**Kalit so‘zlar:** Telegraf tenglamasi, Furyening integral almashtirishi, boshlang‘ich-chegaraviy masala, chegaralanmagan soha, oddiy differensial tenglama.

**Аннотация.** В данной статье исследована задача решения краевой задачи для телеграфного уравнения в первом квартале с использованием метода преобразований Фурье. С помощью косинусного преобразования Фурье задача сведена к обыкновенному дифференциальному уравнению, и для двух физических случаев



(колеблющийся и безколебательный затухающий) получены аналитические решения в явной форме. Доказана единственность решения данной задачи.

**Ключевые слова:** телеграфное уравнение, преобразование Фурье, краевая задача, неограниченная область, обыкновенное дифференциальное уравнение.

**Abstract.** *This article investigates the solution of the initial-boundary value problem for the telegraph equation in the first quarter-plane using the method of Fourier integral transforms. By applying the Fourier cosine transform, the problem is reduced to an ordinary differential equation, and analytical solutions in the form of explicit formulas are obtained for two different physical cases (oscillatory and non-oscillatory damping). The uniqueness of the solution to the given problem is proved.*

**Keywords:** *Telegraph equation, Fourier integral transform, initial-boundary value problem, unbounded domain, ordinary differential equation.*

## INTRODUCTION

When electric current passes through a conductor, an electromagnetic field is generated around it. This field changes the current and voltage in the conductor, causing an oscillatory process. The equation that studies such oscillatory processes is called the Telegraph equation. The Telegraph equation is a mathematical model describing the propagation of electrical signals along transmission lines, simultaneously accounting for wave processes and energy dissipation due to resistance. The second-order time derivative in the equation represents the inertial properties of the signal, while the first-order time derivative accounts for losses in the medium. The spatial derivative describes the propagation of the signal along the cable.

In recent years, many works have been carried out to solve the telegraph equation using various methods. However, solving the initial-boundary value problem in a semi-infinite domain using the Fourier transform is of particular interest. In this article, we investigate the following problem for the telegraph equation in the first quarter-plane:

**Problem.** In the domain  $\Omega = \{(x,t) : 0 < x < +\infty, 0 < t < +\infty\}$ , find a solution of the equation

$$U_{tt} = a^2 U_{xx} + kU, \quad (x,t) \in \Omega \quad (1)$$

satisfying the conditions



$$U_x(0,t) = 0, \quad 0 \leq t < +\infty, \quad (2)$$

$$U(x,0) = f(x), \quad U_t(x,0) = g(x), \quad 0 \leq x \leq +\infty \quad (3)$$

This problem describes the propagation of a signal along a semi-infinite cable. We will find the solution of the problem using the method of Fourier integral transforms. Fourier integral transforms decompose a wave or signal into low-frequency components. For each Fourier component of the differential equation, the frequency-dependent dynamics become simpler, meaning that the given differential equation is reduced to an ordinary differential equation. The general solution is obtained by summing these components, which corresponds to applying the inverse Fourier transform.

As is known, the problem  $\{(1), (2), (3)\}$  is studied in the domain  $(0, +\infty)$ . However, Fourier integral transforms are applied over the entire real axis  $(-\infty, +\infty)$ . Therefore,  $U(x,t)$  can be extended to  $(-\infty, 0)$  as an even function. That is:

$$U(-x,t) = U(x,t).$$

This even extension corresponds to the Fourier cosine transform. In this approach, the general solution  $U(x,t)$  is expressed as the Fourier transform of some function  $\tilde{U}(\lambda,t)$ , namely:

$$\tilde{U}(\lambda,t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} U(x,t) \cos \lambda x dx. \quad (4)$$

According to expression (4), from the inverse Fourier transform, the function  $U(x,t)$  can be reconstructed as:

$$U(x,t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{U}(\lambda,t) \cos \lambda x d\lambda \quad (5)$$

We compute the first and second order partial derivatives of this equality with respect to the variables  $x$  and  $t$ . That is:

$$U_{xx}(x,t) = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \lambda^2 \tilde{U}(\lambda,t) \cos \lambda x d\lambda,$$

$$U_{tt}(x,t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{U}_{tt}(\lambda,t) \cos \lambda x d\lambda.$$



Substituting the above expressions into equation (1), we obtain the following ordinary differential equation:

$$\mathcal{U}_t^{\circ}(\lambda, t) = (k - \lambda^2 a^2) \mathcal{U}^{\circ}(\lambda, t) \quad (6)$$

Now, based on (5), we derive the following:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \mathcal{U}^{\circ}(\lambda, 0) \cos \lambda x d\lambda,$$

$$g(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \mathcal{U}_t^{\circ}(\lambda, 0) \cos \lambda x d\lambda.$$

According to the inverse Fourier transform, we can write the equalities:

$$\mathcal{U}^{\circ}(\lambda, 0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \lambda x dx = f_{\lambda}, \quad (7)$$

$$\mathcal{U}_t^{\circ}(\lambda, 0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(x) \cos \lambda x dx = g_{\lambda}. \quad (8)$$

By introducing  $\omega(\lambda) = k - \lambda^2 a^2$  notation and using expressions (6), (7), and (8), we arrive at the following equation:

$$\begin{cases} \tilde{U}_t(\lambda, t) - \omega(\lambda) \tilde{U}(\lambda, t) = 0 \\ \tilde{U}(\lambda, 0) = f_{\lambda}, \tilde{U}_t(\lambda, 0) = g_{\lambda}. \end{cases} \quad (9)$$

We find the solution of the resulting problem (9) for two cases depending on  $\omega(\lambda)$ .

**Case 1.** Let  $\omega(\lambda) > 0$ . We form the characteristic equation for equation (6):

$$q^2 - \omega(\lambda) = 0$$

The solution of this equation has the form  $q = \pm \omega(\lambda)$ . Consequently, the general solution is expressed as:

$$\mathcal{U}^{\circ}(\lambda, t) = C_1 e^{\omega(\lambda)t} + C_2 e^{-\omega(\lambda)t} \quad (10)$$

Using the conditions  $\mathcal{U}^{\circ}(\lambda, 0) = f_{\lambda}$  and  $\mathcal{U}_t^{\circ}(\lambda, 0) = g_{\lambda}$  from problem (9), we obtain from the general solution (10) a system of equations for the unknowns  $C_1$  and  $C_2$ :

$$\begin{cases} C_1 + C_2 = \tilde{f}(\lambda) \\ \omega(\lambda) C_1 e^{\omega(\lambda)t} - \omega(\lambda) C_2 e^{-\omega(\lambda)t} = \tilde{g}(\lambda) \end{cases}$$



Solving this system, we obtain:

$$C_1 = \frac{g_\lambda}{2\omega(\lambda)} + \frac{f_\lambda}{2}; \quad C_2 = \frac{f_\lambda}{2} - \frac{g_\lambda}{2\omega(\lambda)}.$$

Substituting these into (10), we get:

$$\mathcal{U}(\lambda, t) = \frac{f(\lambda)}{2} \left( e^{\omega(\lambda)t} + e^{-\omega(\lambda)t} \right) + \frac{g(\lambda)}{2\omega(\lambda)} \left( e^{\omega(\lambda)t} - e^{-\omega(\lambda)t} \right).$$

Based on all calculations and the inverse Fourier transform (5), for the case  $\omega(\lambda) > 0$ , the solution of the problem can be written as:

$$U(x, t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \left( \tilde{f}(\lambda) \left( e^{\omega(\lambda)t} + e^{-\omega(\lambda)t} \right) + \frac{\tilde{g}(\lambda)}{\omega(\lambda)} \left( e^{\omega(\lambda)t} - e^{-\omega(\lambda)t} \right) \right) \cos \lambda x d\lambda.$$

This case corresponds to an oscillatory signal whose amplitude decays exponentially over time. This behavior reflects the real physical nature of signals propagating along a cable.

**Case 2.** Now, for the case  $\omega(\lambda) < 0$ , we write the general solution of equation (9):

$$\mathcal{U}(\lambda, t) = C_1 \cos \sqrt{|\omega(\lambda)|}t + C_2 \sin \sqrt{|\omega(\lambda)|}t. \quad (11)$$

According to the conditions of problem (9), we obtain the system:

$$\begin{cases} C_1 = \tilde{f}(\lambda) \\ -C_1 \sqrt{|\omega(\lambda)|} \sin \sqrt{|\omega(\lambda)|}t + C_2 \sqrt{|\omega(\lambda)|} \cos \sqrt{|\omega(\lambda)|}t = \tilde{g}(\lambda). \end{cases}$$

As in the previous case, we find the unknowns  $C_1$  and  $C_2$  in the form:

$$C_1 = f_\lambda; \quad C_2 = \frac{g_\lambda}{\sqrt{|\omega(\lambda)|}}.$$

Substituting the obtained expressions into (11) yields the solution of problem (9) for the case  $\omega(\lambda) < 0$  as follows:

$$U(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left( \tilde{f}(\lambda) \cos \sqrt{|\omega(\lambda)|}t + \frac{\tilde{g}(\lambda)}{\sqrt{|\omega(\lambda)|}} \sin \sqrt{|\omega(\lambda)|}t \right) \cos \lambda x d\lambda.$$

This case represents a non-oscillatory, purely damped process. Such a situation describes a rapid attenuation of the signal when strong resistance is present.



The obtained results demonstrate that Fourier integral transforms are an effective method for analyzing signal propagation using the telegraph equation.

Now we will show the uniqueness of the solution to equation (1) satisfying conditions (2) and (3). Assume that the problem has two distinct solutions  $U_1(x,t)$  and  $U_2(x,t)$ . Their difference is denoted by

$$W(x,t) = U_1(x,t) - U_2(x,t).$$

Then the function  $W(x,t)$  satisfies the following homogeneous problem:

$$\begin{cases} W_{tt} = a^2 W_{xx} + kW, & x > 0, t > 0 \\ W(x,0) = 0, & x \geq 0 \\ W_t(x,0) = 0, & x \geq 0 \\ W(0,t) = 0, & t \geq 0 \end{cases}$$

This problem is similar to the original one, and the calculations performed above remain valid. That is, taking

$$\tilde{W}(\lambda,t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} W(x,t) \cos(\lambda x) dx,$$

we obtain the following problem:

$$\begin{cases} \tilde{W}_{tt} + (a^2 \lambda^2 - k) \tilde{W} = 0 \\ \tilde{W}(\lambda,0) = 0 \quad \tilde{W}_t(\lambda,0) = 0. \end{cases}$$

The corresponding characteristic equation is:

$$r^2 + (a^2 \lambda^2 - k) = 0; \quad r_{1,2} = \pm i \sqrt{a^2 \lambda^2 - k} \quad \text{or} \quad r_{1,2} = \pm \sqrt{k - a^2 \lambda^2}.$$

From this, we can write the general solution:

$$W(\lambda,t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

To determine the unknowns  $C_1$  and  $C_2$  from the initial conditions, we have the system:

$$\begin{cases} C_1 + C_2 = 0 \\ r_1 C_1 + r_2 C_2 = 0. \end{cases}$$



From this system, it follows that  $C_1 = C_2 = 0$  (since the determinant of the system is  $r_1 \neq r_2$ ). Consequently, it is not difficult to determine that  $\tilde{W}(\lambda, t) = 0$ . This implies that  $W(x, t) = 0$  meaning  $U_1(x, t) \equiv U_2(x, t)$ . Thus, the assumption about two distinct solutions is incorrect. In other words, the solution to the initial-boundary value problem for the telegraph equation is unique.

### LITERATURE REVIEW

Previous studies on the topic indicate that the telegraph equation and its related initial-boundary value problems have been extensively studied in mathematical physics. In the works of Zikirov (2017) and Alimov & Ashurov (2017), the equations of mathematical physics, solution methods for ordinary differential equations, derivation of the telegraph equation, and Fourier integral transforms are described in detail.

Russian researchers, including Pikulin & Pokhozaev (2004), developed practical courses for the telegraph equation and other physical equations, where the methodology for solving problems using Fourier and Laplace transforms is explained. Additionally, English sources such as Strauss (2007) and Farlow (1993) provide a scientific analysis of differential equations and demonstrate the application of both analytical and numerical methods.

### RESEARCH METHODOLOGY

In this article, the Fourier integral transform method is applied to solve the initial-boundary value problem for the telegraph equation. The problem is considered in the first quadrant and is reduced to an ordinary differential equation using the Fourier cosine transform. Analytical solutions are obtained in explicit form for two physical cases: oscillatory and non-oscillatory decay. The uniqueness of the solution is mathematically proven.

### CONCLUSION/RECOMMENDATIONS

In this study, the initial-boundary value problem for the telegraph equation was solved using Fourier integral transforms. Analytical solutions were obtained for two physical cases: oscillatory and non-oscillatory decay, and the uniqueness of the solution was proven.



The results can be applied to analyze signal propagation and in telecommunication systems. In future work, this method may be extended to other domains or higher-order equations.

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